

MIXING AND WEAKLY MIXING ABELIAN SUBALGEBRAS OF TYPE II_1 FACTORS

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ABSTRACT. This paper studies weakly mixing (singular) and mixing masas in type II_1 factors from a bimodule point of view. Several necessary and sufficient conditions to characterize the normalizing algebra of a masa are presented. We also study the structure of mixing inclusions, with special attention paid to masas of product class. A recent result of Jolissaint and Stalder concerning mixing masas arising out of inclusions of groups is revisited. One consequence of our structural results rules out the existence of certain Koopman-realizable measures, arising from semidirect products, which are absolutely continuous but not Lebesgue. We also show that there exist uncountably many pairwise non-conjugate mixing masas in the free group factors each with Pukánszky invariant $\{1, \infty\}$.

Keywords: von Neumann algebra; mixing masa; singular masa; measure-multiplicity invariant, Pukánszky invariant

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1. INTRODUCTION

This paper is a continuation of work initiated by the authors in [4], and deals with maximal abelian self-adjoint subalgebras (masas in the sequel) in finite von Neumann algebras. In particular, we study the various notions of singular masas in II_1 factors that arise from notions in ergodic theory and undertake a systematic analysis of the bimodules associated to these masas. While the techniques used in [4] were more operator algebraic in nature, those used in this note are primarily measure-theoretic, so there is little overlap between the techniques deployed here and those found in [4]. It will be evident from the results in §3–§6 that the measure-theoretic approaches in this paper yield new insights into the notion of singularity in the context of masas: among other results, in what follows we relate the mixing properties of masas to those of certain associated measures; build on a number of recent results on strong singularity in [15, 22, 23, 33, 37]; and give a new construction of many non-conjugate mixing masas in the free group factors.

The study of normalizers of masas in II_1 factors has a long history which dates back to 1954 [8]. Let M be a finite von Neumann algebra gifted with a fixed faithful, normal, tracial state τ . Let M be in its standard form i.e., acts on the GNS space associated to τ by left multiplication. Given a masa $A \subset M$, Dixmier defined the group of normalizing unitaries of A to be $N(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$, where $\mathcal{U}(M)$ denotes the unitary group of M [8]. He named A to be singular, if $N(A)$ is as small as possible i.e., $N(A) \subset A$. This definition is purely algebraic but the true nature of singularity was first unveiled by Nielsen [26]. In [26], it was shown that for a (free) ergodic measure preserving action α of a countable discrete abelian group G on a standard probability space, the copy of the group in the associated crossed product produces a singular masa if and only if α is weakly mixing (see [15, 21] for

the relevant definitions).

Note that every separable II_1 factor has singular masas [28]. The authors of [30, 33] studied the notion of ‘infinity–two’ norm to handle singular masas and defined an apparently stronger notion of singularity known as strong singularity. Subsequently, in [37] it was proved that strong singularity is equivalent to singularity and:

Theorem 1.1. [37] *The masa $A \subset M$ is singular if and only if, given any finite set $x_i \in M$ with $\mathbb{E}_A(x_i) = 0$ for all i and for every $\epsilon > 0$, there exists a unitary $u \in A$ such that $\|\mathbb{E}_A(x_i u x_j)\|_2 < \epsilon$ for all i, j , where \mathbb{E}_A denotes the unique trace preserving normal conditional expectation onto A .*

This last property is known as the weak asymptotic homomorphism property (WAHP in the sequel). It was shown in [22, Theorem 6.6] that the unitary in the definition of the WAHP can always be chosen to be v^k , for some positive integer k , where $v \in A$ is a Haar unitary generator of the masa A .

The main topic of this paper – the notion of a strongly mixing masa in a finite von Neumann algebra – was introduced by Jolissaint and Stalder in [15] (see Definition 1.2). They proved that if a countable discrete abelian group G acts on M by (free) mixing τ –preserving automorphisms, then the copy of the group in the associated crossed product produces a strongly mixing masa (in what follows, we will use the terms ‘strongly mixing masa’ and ‘mixing masa’ interchangeably). In [4], the authors gave a formulation of Jolissaint and Stalder’s definition of strong mixing for general subalgebras, and showed that one can replace the groups of unitaries found in the definition of [15] by bounded weakly null sequences. The following definition was shown in [4] to be equivalent to the one in [15], in the context of masas.

Definition 1.2. (Compare with [15, Definition 3.4]) *A masa $A \subset M$ is strongly mixing, if for any bounded sequence of operators $a_n \in A$ converging to zero in the w.o.t (weakly null in the sequel) and $x, y \in M$ such that $\mathbb{E}_A(x) = 0$ and $\mathbb{E}_A(y) = 0$, one has that $\|\mathbb{E}_A(x a_n y)\|_2 \rightarrow 0$ as $n \rightarrow \infty$.*

The outline of the paper is as follows. In Section 2, we will associate to a masa in a II_1 factor a measure class on a compact space of the form $X \times X$, called the left-right measure, as well as a multiplicity function on the same space (see Definition 2.1); these two objects together encode the structure of the standard Hilbert space as a natural bimodule over the masa. Most of the subsequent analysis in the paper will focus on the measure class. In the third section, we present a set of equivalent conditions that characterize the operators in the normalizing algebra of a general masa in a II_1 factor (Theorem 3.2). Crucial to the proof of this result is a classical theorem of Wiener on Fourier coefficients of measures. Theorem 3.7, a main outcome of our analysis in Section 3, is a generalization of the asymptotic homomorphism property introduced in [33]. We show there is a sequence of positive integers k_l such that $\|\mathbb{E}_A(x v^{k_l} x^*)\|_2 \rightarrow 0$ as $l \rightarrow \infty$ for all $x \in M$ such that $\mathbb{E}_{N(A)''}(x) = 0$, where v is a Haar unitary generator of the masa A . These statements are independent of the choice of coordinates, i.e., the choice of the Haar unitary generator.

In Section 4, we study a special class of masas called masas of product class (also studied in [23]), which possess vigorous mixing properties. In particular, they are mixing, (Theorem 4.2), and the convergence in the definition of mixing for masas of product class is the stronger notion of almost everywhere convergence (Theorem 4.4). Our consideration of masas of product class was inspired by a similar, though slightly different, class of masas originating in work

of Sinclair and Smith (cf. [32, §11]). For masas of product class, we demonstrate the existence of sufficiently many vectors $\zeta \in L^2(M) \ominus L^2(A)$ for which $\mathbb{E}_A(\zeta v^n \zeta^*) = 0$ for all $n \neq 0$, where v is a Haar unitary generator of A (Theorem 4.10). A slightly weaker form of the previous statement is a special property for masas in free group factors by a deep theorem of Voiculescu [39]. We also show that a large class of mixing masas in II₁ factors, namely those which arise out of inclusions of groups, fall in the product class (Theorem 4.7). This invigorates one of the main questions addressed in [15, Theorem 3.5]. This result seems to be very interesting, as the combinatorial relations in [15] that determine mixing turn out to be spectral analytic in nature.

The subject of Section 5 is mixing masas that arise from measurable dynamical systems. A Radon measure μ on $[0, 1]$ is said to be mixing (or, sometimes, Rajchman) if its Fourier coefficients $\hat{\mu}_n = \int_0^1 e^{2\pi i n t} d\mu(t)$ converge to zero as $|n| \rightarrow \infty$. One can also define mixing measures on the circle (more generally on separable compact abelian groups) by integrating the functions z^n with respect to the measure. The measure μ is called weak mixing or non-atomic (see [17]), if its Fourier coefficients converge strongly (in absolute value) to zero in the sense of Cesàro. We show that for masas arising out of mixing actions along the direction of the groups in the associated crossed products, the disintegrations of their left-right measures with respect to the coordinate projections yield mixing measures for almost all fibres (Theorem 5.1). This, in turn, justifies the terminology ‘mixing masa.’ We compute the left-right measures of masas arising out of group actions by relating it to the maximal spectral types of the actions (Theorem 5.3). This was stated in [25], but our way of calculating has some advantages; the operators and expressions involved in the definition of WAHP and mixing naturally pop up in our calculation. For standard results about direct integrals used in these analyses, we refer the reader to [18].

The following is an old open problem in ergodic theory (see Remark 5.5): Can the maximal spectral type of an ergodic transformation be absolutely continuous but not Lebesgue? For an excellent account on these class of problems check [16, §6]. Measures (strictly speaking equivalence classes of measures) which arise as maximal spectral type of \mathbb{Z} -systems will be called Koopman-realizable. When Theorem 4.7 is combined with Theorem 5.3, we conclude: There does not exist any countable discrete non abelian group of the form $G \rtimes \mathbb{Z}$ such that

- (i) $L(\mathbb{Z}) \subset L(G \rtimes \mathbb{Z})$ is a mixing masa,
- (ii) the maximal spectral type of the \mathbb{Z} -action on $L(G)$ is absolutely continuous but not Lebesgue (see Corollary 4.9, Corollary 5.6).

Borrowing ideas from ergodic theory, in Section 6 we exhibit examples of uncountably many pairwise non conjugate (by automorphisms) mixing masas in the free group factors each with Pukánszky invariant $\{1, \infty\}$ (Theorem 6.2). Finally, a technical result concerning left-right measures, that is necessary in the later sections of the paper, is proved in the appendix.

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2. PRELIMINARIES AND SETUP

All measure spaces appearing in this paper are assumed to be standard Borel spaces, all von Neumann algebras are assumed to be separable, and all inclusions of algebras are unital. Whereas in [4], we dealt with general inclusions of finite von Neumann algebras, in this paper,

we focus on the special case of masas in II_1 factors. Let M be a II_1 factor, equipped with a faithful, normal, tracial state τ . The trace τ induces an inner product on M by $\langle x, y \rangle = \tau(y^*x)$, $x, y \in M$, and thus induces a Hilbert norm $\|\cdot\|_2$ on M . Let $L^2(M) := L^2(M, \tau)$ denote the completion of M in $\|\cdot\|_2$. We assume that M acts on $L^2(M)$ via left multiplication. Let J denote the conjugation operator on $L^2(M)$, obtained from extending the densely defined map $J : M \subset L^2(M) \rightarrow M \subset L^2(M)$ by $Jx = x^*$. The image of a L^2 -vector ζ under J will be denoted by ζ^* . Let $A \subset M$ be a masa and let $e_A : L^2(M) \rightarrow L^2(A)$ be the Jones projection associated to A , where $L^2(A) = \overline{A}^{\|\cdot\|_2}$. Denote $\mathcal{A} = (A \cup JAJ)''$. It is well known that $e_A \in \mathcal{A}$ (cf. [32]). The one-norm on M is defined as $\|x\|_1 = \tau(|x|)$, $x \in M$. It is also true that $\|x\|_1 = \sup_{y \in M: \|y\| \leq 1} |\tau(xy)|$, $x \in M$. The completion of M in $\|\cdot\|_1$ is denoted by $L^1(M)$. The

τ -preserving normal conditional expectation \mathbb{E}_A onto A , the trace and J extend to $L^1(M)$ in a continuous fashion, and, $L^1(M)$ is also identified with the predual of M . With abuse of notation, we write $e_A(\zeta) = \mathbb{E}_A(\zeta)$ for $L^1(M)$ and $L^2(M)$ vectors. Similarly, we will use the same symbols τ and J to denote their extensions. This will be clear from the context and will cause no confusion. For more details check [32].

Given a type I von Neumann algebra B , write $\text{Type}(B)$ for the set of all those $n \in \mathbb{N} \cup \{\infty\}$, such that B has a nonzero component of type I_n . The Pukánszky invariant of a masa $A \subset M$, denoted by $\text{Puk}(A)$ (or $\text{Puk}_M(A)$ when the containing factor is ambiguous) is defined to be $\text{Type}(\mathcal{A}'(1 - e_A))$ [29]. A stronger invariant for masas in II_1 factors called the measure-multiplicity invariant was studied in [12, 22, 23, 25], and was used in [12, 23] to distinguish masas with same Pukánszky invariant [29]. The measure-multiplicity invariant of a masa has two main components: a measure class (*left-right measure*) and a multiplicity function, which together encode the structure of the standard Hilbert space $L^2(M)$ as the natural A, A -bimodule. Both are spectral invariants. Such study of bimodules first appeared in a handwritten notes of Connes [6].

For a masa $A \subset M$, one fixes a unital, norm separable and σ -weakly dense (also dense in the *w.o.t*) C^* -subalgebra of A which is isomorphic to $C(X)$ for some compact metric (Hausdorff) space X . Let λ denote the tracial measure on X . For $\zeta_1, \zeta_2 \in L^2(M)$, let $\kappa_{\zeta_1, \zeta_2} : C(X) \otimes C(X) \rightarrow \mathbb{C}$ be the linear functional defined by,

$$\kappa_{\zeta_1, \zeta_2}(a \otimes b) = \langle a\zeta_1 b, \zeta_2 \rangle, \quad a, b \in C(X).$$

Then $\kappa_{\zeta_1, \zeta_2}$ induces a unique complex Radon measure η_{ζ_1, ζ_2} on $X \times X$ given by,

$$(1) \quad \kappa_{\zeta_1, \zeta_2}(a \otimes b) = \int_{X \times X} a(t)b(s) d\eta_{\zeta_1, \zeta_2}(t, s),$$

and $\|\eta_{\zeta_1, \zeta_2}\|_{t.v} = \|\kappa_{\zeta_1, \zeta_2}\|$, where $\|\cdot\|_{t.v}$ denotes the total variation norm of measures.

Write $\eta_{\zeta, \zeta} = \eta_\zeta$. Note that η_ζ is a positive measure for all $\zeta \in L^2(M)$. It is easy to see that the following polarization type identity holds:

$$(2) \quad 4\eta_{\zeta_1, \zeta_2} = (\eta_{\zeta_1 + \zeta_2} - \eta_{\zeta_1 - \zeta_2}) + i(\eta_{\zeta_1 + i\zeta_2} - \eta_{\zeta_1 - i\zeta_2}).$$

Note that the decomposition of η_{ζ_1, ζ_2} in equation (2) need not be its Hahn decomposition in general, but

$$(3) \quad 4|\eta_{\zeta_1, \zeta_2}| \leq (\eta_{\zeta_1 + \zeta_2} + \eta_{\zeta_1 - \zeta_2}) + (\eta_{\zeta_1 + i\zeta_2} + \eta_{\zeta_1 - i\zeta_2}) = 4(\eta_{\zeta_1} + \eta_{\zeta_2}),$$

so that

$$(4) \quad |\eta_{\zeta_1, \zeta_2}| \leq \eta_{\zeta_1} + \eta_{\zeta_2}.$$

For a set X , denote by $\Delta(X)$ the diagonal of $X \times X$.

Definition 2.1. [12, 22, 25] *The measure–multiplicity invariant of $A \subset M$, denoted by $m.m(A)$, is the equivalence class of quadruples $(X, \lambda_X, [\eta|_{\Delta(X)^c}], m|_{\Delta(X)^c})$ under the equivalence relation $\sim_{m.m}$, where*

(i) X is a compact Hausdorff space such that $C(X)$ is a unital, norm separable, w.o.t dense subalgebra of A ,

(ii) λ_X is the Borel probability measure obtained by restricting the trace τ on $C(X)$, and

(iii) η is the measure on $X \times X$ and

(iv) m is the multiplicity function,

both obtained from the direct integral decomposition of $L^2(M)$, so that \mathcal{A} is the algebra of diagonalizable operators with respect to this decomposition; the equivalence $\sim_{m.m}$ being,

$$(X, \lambda_X, [\eta|_{\Delta(X)^c}], m|_{\Delta(X)^c}) \sim_{m.m} (Y, \lambda_Y, [\eta|_{\Delta(Y)^c}], m|_{\Delta(Y)^c})$$

if and only if, there exists a Borel isomorphism $F : X \rightarrow Y$, such that

$$F_*\lambda_X = \lambda_Y,$$

$$(F \times F)_*[\eta|_{\Delta(X)^c}] = [\eta|_{\Delta(Y)^c}],$$

$$m|_{\Delta(X)^c} \circ (F \times F)^{-1} = m|_{\Delta(Y)^c}, \quad \eta|_{\Delta(Y)^c} \text{ a.e.}$$

It is easy to see that the Pukánszky invariant of $A \subset M$ is the set of essential values of the multiplicity function in Definition 2.1. The measure class $[\eta|_{\Delta(X)^c}]$ is said to be the **left–right measure** of A . Both $m.m(\cdot)$ and $Puk(\cdot)$ are invariants of the masa under automorphisms of the factor M . In this paper, we are mostly interested in the *left–right measure*.

To understand the relation between mixing properties of masas and their left–right measures, disintegration of measures will be used (for a detailed account of disintegration, consult [5]). Let T be a measurable map from (X, σ_X) to (Y, σ_Y) , where σ_X, σ_Y are σ –algebras of subsets of X, Y respectively. Let β be a σ –finite measure on σ_X and μ a σ –finite measure on σ_Y . Here β is the measure to be disintegrated and μ is often the push forward measure $T_*\beta$, although other possibilities for μ are allowed.

Definition 2.2. *We say that β has a disintegration $\{\beta^t\}_{t \in Y}$ with respect to T and μ or a (T, μ) –disintegration if:*

(i) β^t is a σ –finite measure on σ_X concentrated on $\{T = t\}$ (or $T^{-1}\{t\}$), i.e., $\beta^t(\{T \neq t\}) = 0$ for μ –almost all t ,

and, for each nonnegative σ_X –measurable function f on X :

(ii) $t \mapsto \beta^t(f)$ is σ_Y –measurable.

(iii) $\beta(f) = \mu^t(\beta^t(f)) \stackrel{\text{defn}}{=} \int_Y \beta^t(f) d\mu(t)$.

If β in Definition 2.2 is a complex measure, then a disintegration of β is obtained by first decomposing it into a linear combination of four positive measures, using the Hahn decomposition of its real and imaginary parts. Given a measure λ on X and coordinate projections $\pi_i : X \times X \rightarrow X$, $i = 1, 2$, we will index the (π_1, λ) – and (π_2, λ) –disintegrations of a measure on $X \times X$ using superscripts of t and s , respectively. In particular, we will make use of the disintegrations $\{\eta_\zeta^t\}_t$ and $\{\eta_\zeta^s\}_s$ of the measures η_{ζ_1, ζ_2} defined by equation (1); these disintegrations are known to exist by [22, Theorem 3.2] (see also [5]). Note that the measure η_ζ^t (respectively, η_ζ^s) is concentrated on $\{t\} \times X$ (respectively, $X \times \{s\}$) for λ –almost every

t (respectively, λ -almost every s). Denote the restriction of η_ζ^t to $\{t\} \times X$ by $\tilde{\eta}_\zeta^t$, and the restriction of η_ζ^s to $X \times \{s\}$ by $\tilde{\eta}_\zeta^s$.

The left–right measure of the masa A has the property that if $\theta : X \times X \rightarrow X \times X$ is the flip map $\theta(t, s) = (s, t)$, then $\theta_*\eta \ll \eta \ll \theta_*\eta$ [22, Lemma 2.9]. In fact, it is possible to obtain a choice of η for which $\theta_*\eta = \eta$. Therefore, in most of the following, we will only state or prove results with respect to the (π_1, λ) –disintegration; analogous results with respect to the (π_2, λ) –disintegration are also possible. The following lemma, a proof of which can be found in [22, §6] and [23], will be crucial for our purposes.

Lemma 2.3. *Let $\zeta_1, \zeta_2 \in L^2(M)$ be such that $\mathbb{E}_A(\zeta_1) = 0 = \mathbb{E}_A(\zeta_2)$. Let η_{ζ_1, ζ_2} denote the Borel measure on $X \times X$ defined in equation (1).*

1°. *Then η_{ζ_1, ζ_2} admits (π_i, λ) –disintegrations $X \ni t \mapsto \eta_{\zeta_1, \zeta_2}^t$ and $X \ni s \mapsto \eta_{\zeta_1, \zeta_2}^s$. Moreover,*

$$\eta_{\zeta_1, \zeta_2}^t(X \times X) = \mathbb{E}_A(\zeta_1 \zeta_2^*)(t), \quad \lambda \text{ a.e.}$$

2°. *Let $f \in C(X)$. Then the functions $X \ni t \mapsto \eta_{\zeta_1, \zeta_2}^t(1 \otimes f)$, $X \ni s \mapsto \eta_{\zeta_1, \zeta_2}^s(f \otimes 1)$ are in $L^1(X, \lambda)$. If $\zeta_i \in M$ for $i = 1, 2$, then $X \ni t \mapsto \eta_{\zeta_1, \zeta_2}^t(1 \otimes f)$, $X \ni s \mapsto \eta_{\zeta_1, \zeta_2}^s(f \otimes 1)$ are in $L^\infty(X, \lambda)$.*

3°. *Let $b, w \in C(X)$. If $\mathbb{E}_A(\zeta_1 w \zeta_2^*) \in L^2(A)$, then*

$$\begin{aligned} \|\mathbb{E}_A(b \zeta_1 w \zeta_2^*)\|_2^2 &= \int_X |b(t)|^2 |\eta_{\zeta_1, \zeta_2}^t(1 \otimes w)|^2 d\lambda(t), \\ \|\mathbb{E}_A(b \zeta_1 w \zeta_2^*)\|_1 &= \int_X |b(t)| |\eta_{\zeta_1, \zeta_2}^t(1 \otimes w)| d\lambda(t). \end{aligned}$$

3. WEAK MIXING AND THE NORMALIZING ALGEBRA

In this section, we use measure-theoretic techniques to establish several equivalent analytical conditions that characterize the normalizing algebra of a masa. These results, along with those in §5, highlight the relations between mixing, weak mixing of masas and Fourier coefficients of mixing and non-atomic measures. The main ingredients in the characterization are Theorems 5.5, 6.6 of [22] and the following result of Wiener.

Theorem 3.1. (Wiener) *Let μ be a finite Borel measure on S^1 and let $\hat{\mu}(n)$, $n \in \mathbb{Z}$, be its Fourier coefficients, i.e., $\hat{\mu}(n) = \int_{S^1} t^n d\mu(t)$, $n \in \mathbb{Z}$. Then*

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |\hat{\mu}(n)|^2 = \sum_{t \in S^1: \mu(\{t\}) \neq 0} \mu(\{t\})^2.$$

The proof is a direct consequence of the dominated convergence theorem [21, Lemma 1.1].

If M is a fixed II_1 factor and $A \subset M$ a masa, let λ denote the normalized Haar measure on S^1 so that $A \cong L^\infty(S^1, \lambda)$; then λ is the tracial measure. Let $[\eta]$ denote the left–right measure of A . We assume that η is a probability measure on $S^1 \times S^1$, with $\eta(\Delta(S^1)) = 0$. Occasionally in subsequent sections it will be convenient instead to view A as isomorphic to $L^\infty([0, 1], \lambda)$, where λ is Lebesgue measure on $[0, 1]$ (so that η would then be a probability on $[0, 1] \times [0, 1]$). We will notify the reader in context of any such change.

Theorem 3.2. *Let $A \subset M$ be a masa. Let $v \in A$ be the Haar unitary generator corresponding to the function $S^1 \ni t \mapsto t \in S^1$. Let $x \in M$ be such that $\mathbb{E}_A(x) = 0$. Then the following are*

equivalent.

$$(i) \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(xv^k x^*)\|_2^2 = 0.$$

$$(i)' \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(xv^k x^*)\|_2 = 0.$$

$$(ii) \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(xv^k x^*)\|_1^2 = 0.$$

$$(ii)' \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(xv^k x^*)\|_1 = 0.$$

(iii) Given any finite set $\{w_i\}_{i=1}^k \subset \mathcal{U}(A)$, there is a sequence of unitaries $u_n \in A$, such that $\lim_{n \rightarrow \infty} \|\mathbb{E}_A(xw_i u_n x^*)\|_2 = 0$ for all $1 \leq i \leq k$.

(iii)' Given any finite set $\{w_i\}_{i=1}^k \subset \mathcal{U}(A)$, there is a sequence of unitaries $u_n \in A$, such that $\lim_{n \rightarrow \infty} \|\mathbb{E}_A(xw_i u_n x^*)\|_1 = 0$ for all $1 \leq i \leq k$.

$$(iv) \mathbb{E}_{N(A)''}(x) = 0.$$

Proof. That $(i) \Leftrightarrow (i)'$, $(ii) \Leftrightarrow (ii)'$ hold, follows from the fact that whenever $\{a_k\}_{k \in \mathbb{Z}} \subset \mathbb{C}$ is bounded, we have

$$(5) \quad \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N |a_k| = 0 \Leftrightarrow \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N |a_k|^2 = 0.$$

Again, $(iii) \Rightarrow (iii)'$ and $(i) \Rightarrow (ii)$ are obvious as $\|\cdot\|_1$ is dominated by $\|\cdot\|_2$. Also, $(iii)' \Rightarrow (iii)$ follows from Lemma 2.3 after choosing $u_n \in C(S^1)$ (by making a density argument) and passing to a subsequence. $(iv) \Rightarrow (i)$, $(iv) \Rightarrow (ii)$ are direct consequences of Theorem 5.5, Theorem 6.6 and Theorem 6.9 of [22], after replacing the base space by S^1 . So, we have to prove $(ii) \Rightarrow (iv)$ and $(iii) \Leftrightarrow (i)$.

$(ii) \Rightarrow (iv)$. Note that $(ii)'$ holds. Write

$$a_N = \frac{1}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(xv^k x^*)\|_1, \quad b_N = \frac{1}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(xv^k x^*)\|_2^2, \quad \forall N \in \mathbb{N}.$$

Note that a_N and b_N , $N = 1, 2, \dots$ define bounded sequences. There is a constant $C > 0$ and a set N_0 with $\lambda(N_0) = 0$ such that $|\eta_x^t(1 \otimes v^k)| \leq C$ for all $t \in N_0^c$ and all $k \in \mathbb{Z}$.

We claim that given any subsequence b_{N_i} , there is a further subsequence $b_{N_{i_m}}$ such that $b_{N_{i_m}} \rightarrow 0$ as $m \rightarrow \infty$. Then b_N would converge to 0 as $N \rightarrow \infty$. Assume that the claim is true.

Write $x = y + z$, where $y = \mathbb{E}_{N(A)''}(x)$ and $z = x - \mathbb{E}_{N(A)''}(x)$. For $a, b \in A$, one has $\langle ayb, z \rangle = 0$ and $\langle azb, y \rangle = 0$, since $AyA \subseteq N(A)''$. It follows that $\eta_x = \eta_y + \eta_z$. Hence, from

Lemma 3.4 of [22], $\eta_x^t = \eta_y^t + \eta_z^t$ for λ almost all t . Write

$$c_N = \frac{1}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(zv^k z^*)\|_2^2, \quad d_N = \frac{1}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(yv^k y^*)\|_2^2, \quad \forall N \in \mathbb{N}.$$

Thus, from Lemma 2.3,

$$\begin{aligned} |b_N - d_N| &\leq \left| c_N + \frac{2}{2N+1} \sum_{k=-N}^N \int_{S^1} \Re(\eta_z^t(1 \otimes v^k) \eta_y^t(1 \otimes v^{-k})) d\lambda(t) \right| \\ &\leq c_N + \frac{2}{2N+1} \sum_{k=-N}^N \int_{S^1} |\eta_z^t(1 \otimes v^k) \eta_y^t(1 \otimes v^{-k})| d\lambda(t) \\ &\leq c_N + \frac{2}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(zv^k z^*)\|_2 \|\mathbb{E}_A(yv^k y^*)\|_2 \\ &\leq (1 + 2\|y\|^2) c_N. \end{aligned}$$

Consequently as $\mathbb{E}_{N(A)''}(z) = 0$, the hypothesis on x and Theorem 6.9 of [22] (with $[0, 1]$ replaced by S^1) force that

$$\limsup_N d_N = 0,$$

which is a contradiction to Wiener's theorem (Theorem 3.1) unless $y = 0$, from Theorem 5.5 [22]. Therefore, $y = 0$. Thus, we only have to prove the claim.

Fix a subsequence b_{N_l} . Note that $a_{N_l} \rightarrow 0$ as $l \rightarrow \infty$. So by Lemma 2.3 it follows that

$$\int_{S^1} \frac{1}{2N_l+1} \sum_{k=-N_l}^{N_l} |\eta_x^t(1 \otimes v^k)| d\lambda(t) \rightarrow 0, \quad \text{as } l \rightarrow \infty.$$

Dropping to a subsequence $b_{N_{l_m}}$, replacing the null set N_0 by a (probably) larger null set if necessary and renaming it to be N_0 again, it follows that

$$\frac{1}{2N_{l_m}+1} \sum_{k=-N_{l_m}}^{N_{l_m}} |\eta_x^t(1 \otimes v^k)| \rightarrow 0, \quad \text{as } m \rightarrow \infty \text{ for all } t \in N_0^c.$$

Now for $t \in N_0^c$,

$$\frac{1}{2N_{l_m}+1} \sum_{k=-N_{l_m}}^{N_{l_m}} |\eta_x^t(1 \otimes v^k)|^2 \leq C \frac{1}{2N_{l_m}+1} \sum_{k=-N_{l_m}}^{N_{l_m}} |\eta_x^t(1 \otimes v^k)|, \quad \text{for all } m.$$

Direct application of dominated convergence theorem and Lemma 2.3 shows that $b_{N_{l_m}} \rightarrow 0$ as $m \rightarrow \infty$.

(iii) \Rightarrow (i). By making a density argument, we can assume that there is a sequence of unitaries $u_n \in C(S^1) \subset A$, such that $\|\mathbb{E}_A(xw_i u_n x^*)\|_2 \rightarrow 0$ as $n \rightarrow \infty$ for all $1 \leq i \leq k$. We will only show that $\tilde{\eta}_x^t$ is non-atomic for λ almost all t . Then, in view of Theorem 5.5 and Theorem 6.6

of [22], the arguments are complete.

For each $l \in \mathbb{N}$, choose a unitary $u_l \in C(S^1)$, such that

$$\|\mathbb{E}_A(xv^j u_l x^*)\|_2 < \frac{1}{l+1}, \quad j = 0, \pm 1, \dots, \pm l.$$

Lemma 2.3 yields

$$\|\mathbb{E}_A(xv^j u_l x^*)\|_2^2 = \int_{S^1} |\eta_x^t(1 \otimes v^j u_l)|^2 d\lambda(t) < \frac{1}{(l+1)^2}, \quad j = 0, \pm 1, \dots, \pm l, \text{ for all } l.$$

Therefore,

$$\lim_l \int_{S^1} |\eta_x^t(1 \otimes v^j u_l)|^2 d\lambda(t) = 0, \quad \text{for } j = 0, \pm 1, \pm 2, \dots, \pm N, \text{ for all } N \in \mathbb{N}.$$

Using Cantor's diagonal argument, we may extract a subsequence $l_p < l_{p+1}$ for all p , and a set $F \subset S^1$ with $\lambda(F) = 0$, such that for all $t \in F^c$,

$$\lim_p \int_{S^1} s^j u_{l_p}(s) d\tilde{\eta}_x^t(s) = 0$$

for $j = 0, \pm 1, \pm 2, \dots$, and $\tilde{\eta}_x^t$ is a finite measure (see Lemma 2.3). Consequently, by the Stone–Weierstrass theorem, we have for all $f \in C(S^1)$,

$$(6) \quad \lim_p \int_{S^1} f(s) u_{l_p}(s) d\tilde{\eta}_x^t(s) = 0, \quad \text{for } t \in F^c.$$

A further density argument establishes that equation (6) holds if f is the indicator function of a compact set. It follows that $\tilde{\eta}_x^t$ cannot have any atoms for $t \in F^c$.

(i) \Rightarrow (iii). As (i) \Leftrightarrow (iv) so $\mathbb{E}_{N(A)''}(x) = 0$, and hence $\mathbb{E}_{N(A)''}(xw_i) = 0$ for all $1 \leq i \leq k$. Thus $\tilde{\eta}_{xw_i}^t$ and $\tilde{\eta}_x^t$ are non-atomic for all $1 \leq i \leq k$ and for λ almost all t . Use equation (2) to conclude that $\frac{1}{2N+1} \sum_{k=-N}^N |\eta_{xw_i, x}^t(1 \otimes v^k)|^2$ goes to zero as $N \rightarrow \infty$ almost everywhere λ for all $1 \leq i \leq k$. Now use Lemma 2.3 to conclude that $\frac{1}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(xw_i v^k x^*)\|_2^2 \rightarrow 0$ for all i . Thus, there is a set $S \subset \mathbb{Z}$ of density one such that $\|\mathbb{E}_A(xw_i v^k x^*)\|_2$ goes to zero as $|k| \rightarrow \infty$ along S [22]. This completes the proof. \square

Remark 3.3. Note that (iii) \Rightarrow (iv) in Theorem 3.2 is to be compared with Lemma 2.5 in [27].

Corollary 3.4. (*Independence of coordinates*) Let $A \subset M$ be a masa. Let $v \in A$ be the Haar unitary generator corresponding to the function $S^1 \ni t \mapsto t \in S^1$. Let $x \in M$ be such that $\mathbb{E}_A(x) = 0$. Then the following are equivalent.

- (i) $\lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(xv^k x^*)\|_2^2 = 0$.
- (ii) $\lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N \|\mathbb{E}_A(xw^k x^*)\|_2^2 = 0$, for any Haar unitary generator w of A .

Remark 3.5. Note that (i) in Theorem 3.2 is false for any Haar unitary of A . There can be diffuse subalgebras inside A with large normalizers. For example, consider the masa $A(1)$ in [36]. The averaging conditions in Theorem 3.2 are the analogues of weakly mixing actions.

Remark 3.6. Rigidity and non-recurrence are two properties of dynamical systems that are in some sense opposite to each other [3]. When translated to the language of operator algebras, rigidity characterizes masas having non trivial central sequences, while non-recurrence is close to mixing. The sequences along which these properties occur for weakly mixing transformations have rich structure [3]. Thus, in view of the results in [3], it is important to know more of the asymptotic properties of $\mathbb{E}_A(xv^kx^*)$, $k \in \mathbb{Z}$. The operators $\mathbb{E}_A(xv^kx^*)$, $k \in \mathbb{Z}$, are directly related to Fourier coefficients of certain measures that characterize (weak) mixing and rigidity (see proof of Theorem 5.1).

Theorem 3.7. *Let $A \subset M$ be a masa and let $v \in A$ be the Haar unitary generator of A corresponding to the function $S^1 \ni t \mapsto t \in S^1$. There is a subsequence k_l ($k_l < k_{l+1}$) such that $\|\mathbb{E}_A(xv^{k_l}x^*)\|_2 \rightarrow 0$ as $l \rightarrow \infty$ for all $x \in M$ such that $\mathbb{E}_{N(A)''}(x) = 0$.*

Proof. The proof follows easily from Theorem 3.2 by separability and a diagonalization argument. We omit the details. \square

Remark 3.8. When compared with the asymptotic homomorphism property (AHP) introduced in [33], and Proposition 1.1 of [15], we suspect that Theorem 3.7 may be the best result along these lines that can be expected in general. Moreover, by making an argument appealing to Corollary 3.4 and Theorem 3.2, it can be shown that the statement in Theorem 3.7 (except possibly the subsequence) is independent of the choice of Haar unitary generator. In other words, any singular masa ‘almost has the AHP’ with respect to any choice of its Haar unitary generator.

4. MASAS OF PRODUCT CLASS

A masa A in a II_1 factor M is said to be of product class if its left-right measure is the class of product measure. This condition was shown in [23] to be equivalent to the condition that the space $L^2(M) \ominus L^2(A)$ decomposes as a direct sum of coarse A – A bimodules. Masas with this property were studied in detail in [23], though in essence they have been known about for some time. In this section, we study masas of product class in the context of mixing properties of subalgebras. One of our main results in this section builds on those obtained by Jolissaint and Stalder in [15], in particular, [15, Theorem 3.5]. Furthermore, in restricting ourselves to a smaller class of dynamical systems – namely, the ones that arise from semidirect products of groups – the absence of \mathbb{Z} –systems with strictly absolutely continuous spectrum will be a consequence of the analysis undertaken in §4 and §5. We begin by recalling the following property of masas from [23], which is closely related to masas of product class.

Definition 4.1. *We say that a masa $A \subset M$ has the property (SU) if it satisfies the following conditions: There exists a set $S \subset M$ such that $\mathbb{E}_A(x) = 0$ for all $x \in S$, and the linear span of S is dense in $L^2(M) \ominus L^2(A)$; there is an orthonormal basis $\{v_n\}_{n=1}^\infty \subset A$ of $L^2(A)$ such that*

$$(7) \quad \sum_{n=1}^{\infty} \|\mathbb{E}_A(xv_nx^*)\|_2^2 < \infty \text{ for all } x \in S;$$

and there is a nonzero vector $\zeta \in L^2(M) \ominus L^2(A)$ such that $\mathbb{E}_A(\zeta u^n \zeta^) = 0$ for all $n \neq 0$, where u is a Haar unitary generator of A .*

There are many examples of masas in both the hyperfinite and free group factors known to satisfy (SU). In [23], it was shown that any masa satisfying (SU) is also a masa of product

class. In the same work, it was shown that masas of product class satisfy a similar (but slightly weaker) set of conditions to those in (SU) [23, Theorem 2.5]. In the analysis that follows, we assume $A = L^\infty([0, 1], \lambda)$. Note that the results below use Lemma A.1 from the appendix.

Theorem 4.2. *Let $A \subset M$ be a masa of product class. Then A is a mixing masa.*

Proof. In Theorem 2.5 [23], it was shown that the hypothesis implies the following. There is a set $S \subset L^2(M) \ominus L^2(A)$ such that $\text{span } S$ is dense in $L^2(M) \ominus L^2(A)$,

$$\sum_{n \in \mathbb{Z}} \|\mathbb{E}_A(\zeta v^n \zeta^*)\|_2^2 < \infty \text{ for all } \zeta \in S,$$

and there is a nonzero $\xi_0 \in L^2(M) \ominus L^2(A)$ such that $\mathbb{E}_A(\xi_0 v^n \xi_0^*) = 0$ for all $n \neq 0$, where v is the Haar unitary generator of A corresponding to the function $[0, 1] \ni t \mapsto e^{2\pi i t}$. Furthermore, the proof of the same theorem shows that S can be chosen so that $\frac{d\eta_\zeta}{d(\lambda \otimes \lambda)}$ is essentially bounded for $\zeta \in S$.

Use Lemma A.1 and Remark A.2 to conclude that there is a dense subset $S' \subset L^2(M) \ominus L^2(A)$ such that

$$\sum_{n \in \mathbb{Z}} \|\mathbb{E}_A(\xi_1 v^n \xi_2^*)\|_2^2 < \infty \text{ for all } \xi_1, \xi_2 \in S',$$

and $\frac{d\eta_\xi}{d(\lambda \otimes \lambda)}$ is essentially bounded for all $\xi \in S'$. In the above statements, it is implicit that the vectors $\zeta, \xi_i, i = 1, 2$, are such that $\mathbb{E}_A(\zeta v^n \zeta^*), \mathbb{E}_A(\xi_1 v^n \xi_2^*) \in L^2(A)$; thus there is no confusion in considering their L^2 -norms.

Making arguments as in [32, Section 11.4], it is easy to see that if $\{a_n\} \subset A$ is a bounded weakly null sequence of operators, then $\mathbb{E}_A(\xi_1 a_n \xi_2^*) \rightarrow 0$ in $\|\cdot\|_2$ for all $\xi_1, \xi_2 \in S'$.

Fix $x \in M$ such that $\mathbb{E}_A(x) = 0$. Also fix $\xi \in S'$ and a weakly null sequence of operators $\{a_n\} \subset A$ in the unit ball. Choose a sequence of vectors $\zeta_k \in S'$ such that $\zeta_k \rightarrow x$ in $\|\cdot\|_2$. For $\xi \in S'$, from Lemma 2.3 we have $\mathbb{E}_A(\xi \xi^*)(t) = \eta_\xi^t([0, 1] \times [0, 1]) = \int_0^1 f_\xi(t, s) d\lambda(s)$ for λ almost all t , where $f_\xi = \frac{d\eta_\xi}{d(\lambda \otimes \lambda)}$. Since f_ξ is essentially bounded, it follows from Lemma 3.6 [22] that $\mathbb{E}_A(\xi \xi^*) \in L^\infty([0, 1], \lambda)$. Thus, for $n \in \mathbb{N}$,

$$\begin{aligned} \sup_{a \in C[0, 1]: \|a\|_2 \leq 1} \left| \int_0^1 a(t) \mathbb{E}_A((\zeta_k - x) a_n \xi^*)(t) d\lambda(t) \right| &= \sup_{a \in C[0, 1]: \|a\|_2 \leq 1} |\tau((\zeta_k - x) a_n \xi^* a)| \\ &= \sup_{a \in C[0, 1]: \|a\|_2 \leq 1} |\langle (\zeta_k - x) a_n, a^* \xi \rangle| \\ &\leq \|\zeta_k - x\|_2 \sup_{a \in C[0, 1]: \|a\|_2 \leq 1} \langle a^* \xi, a^* \xi \rangle^{\frac{1}{2}} \\ &= \|\zeta_k - x\|_2 \sup_{a \in C[0, 1]: \|a\|_2 \leq 1} \tau(a^* \mathbb{E}_A(\xi \xi^*) a)^{\frac{1}{2}} \\ &\leq \|\mathbb{E}_A(\xi \xi^*)\|^{\frac{1}{2}} \|\zeta_k - x\|_2. \end{aligned}$$

This shows that $\mathbb{E}_A((\zeta_k - x) a_n \xi^*) \in L^2(A)$ for all k, n , and a triangle inequality argument shows that $\mathbb{E}_A(x a_n \xi^*) \rightarrow 0$ in $\|\cdot\|_2$. Make a further density argument to finish the proof. We omit the details. \square

Remark 4.3. A similar argument along with Theorem 3.1 [35] gives a proof of the fact that the radial (Laplacian) masa in $L(\mathbb{F}_k)$, $2 \leq k < \infty$ is mixing. The same can be deduced as a

corollary of Theorem 4.2, as the left–right measure of the radial masa is the class of product measure [11].

The next result shows that masas of product class can in fact be compared directly with mixing masas, and possess far stronger convergence properties.

Theorem 4.4. *Let $A \subset M$ be a masa. Suppose the left–right measure of A is the class of product measure. Let $x, y \in M$ be such that $\mathbb{E}_A(x) = 0 = \mathbb{E}_A(y)$. If (u_n) is a bounded sequence in A converging to zero in the weak operator topology, then $\mathbb{E}_A(xu_ny^*)$ converges to zero λ almost everywhere.*

Before we prove Theorem 4.4, we need to make an observation. Let $x \in M$ be such that $\mathbb{E}_A(x) = 0$. In the results of the third author in [22, 23] that involved disintegration of measures, it was necessary to work with functions of the form $[0, 1] \ni t \mapsto \eta_x^t(1 \otimes a)$, where $a \in C[0, 1] \subset A$ (or $a \in C(S^1) \subset A$ as the case may be). The reason for the choice of $a \in C[0, 1]$ (or $a \in C(S^1)$) in that work was to ensure that the function $[0, 1] \ni t \mapsto \eta_x^t(1 \otimes a)$ was finite almost everywhere and measurable. However, if $[\eta] = [\lambda \otimes \lambda]$, then we can allow a to be in $L^\infty([0, 1], \lambda)$ (or $L^\infty(S^1, \lambda)$). In this case, the aforementioned finiteness and measurability are not issues.

Proof of Theorem 4.4. First, fix $x \in M$ with $\mathbb{E}_A(x) = 0$. Note that $\eta_x \ll \lambda \otimes \lambda$ [12, Lemma 5.7]. Let $g = \frac{d\eta_x}{d(\lambda \otimes \lambda)}$. Then $g \in L^1(\lambda \otimes \lambda)$. From Lemma 3.6 [22], $\tilde{\eta}_x^t \ll \lambda$ and $\frac{d\tilde{\eta}_x^t}{d\lambda} = g_t$ for λ almost all t , where $g_t = g(t, \cdot)$.

It is easy to verify that, $[0, 1] \ni t \mapsto \eta_x^t(1 \otimes u_n)$ is in $L^\infty([0, 1], \lambda)$ for all n (use Lemma 2.3). For $a \in C[0, 1]$, the equation

$$(8) \quad \langle \mathbb{E}_A(xu_nx^*), a \rangle = \tau(a^* \mathbb{E}_A(xu_nx^*)) = \tau(a^*xu_nx^*) = \int_0^1 \overline{a(t)} \eta_x^t(1 \otimes u_n) d\lambda(t)$$

implies that, $\mathbb{E}_A(xu_nx^*)(t) = \eta_x^t(1 \otimes u_n)$ for λ almost all t . Thus, for λ almost all t we have,

$$\mathbb{E}_A(xu_nx^*)(t) = \eta_x^t(1 \otimes u_n) = \int_0^1 u_n(s) g_t(s) d\lambda(s) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The last statement holds because $\{u_n\}$ is bounded, converges to zero in *w.o.t* and $g_t \in L^1(\lambda)$ for almost all t .

Finally use the identity,

$$\begin{aligned} 4 \mathbb{E}_A(xu_ny^*) &= \mathbb{E}_A((x+y)u_n(x+y)^*) - \mathbb{E}_A((x-y)u_n(x-y)^*) \\ &\quad + i \mathbb{E}_A((x+iy)u_n(x+iy)^*) - i \mathbb{E}_A((x-iy)u_n(x-iy)^*), \text{ for all } n, \end{aligned}$$

to complete the proof. \square

Remark 4.5. Observe that in Theorem 4.4, even if we assume only that the left–right measure is absolutely continuous with respect to the product measure, then the conclusions remain valid.

The following lemma is likely to be well-known to experts, but we lack a reference, so we present it here for convenience.

Lemma 4.6. *Let H be a countable discrete torsion-free abelian group. Let G be a closed subgroup of \hat{H} such that the normalized Haar measure λ_G of G is absolutely continuous with*

respect to the normalized Haar measure $\lambda_{\widehat{H}}$ of \widehat{H} (regarding λ_G to be a measure on \widehat{H} by extending it by zero on the complement of G). Then $G = \widehat{H}$.

Proof. Note that both \widehat{H} and G are compact abelian groups, thus λ_G and $\lambda_{\widehat{H}}$ exist. Consequently, $\lambda_{\widehat{H}}(G) > 0$. Since H is torsion-free, so \widehat{H} is connected (see Theorem 24.25 [14]). Since $\lambda_{\widehat{H}}$ is normalized, the translation invariance of Haar measure forces that \widehat{H}/G is a finite abelian group. However, the quotient map $q : \widehat{H} \rightarrow \widehat{H}/G$ is continuous, so the image of q is connected. Thus, \widehat{H}/G is trivial. \square

Theorem 4.7. *Let Γ be a countable discrete non abelian group and let Γ_0 be an infinite abelian subgroup of Γ . Suppose $L(\Gamma_0) \subset L(\Gamma)$ is a mixing masa. Then,*

- (i) *the left–right measure of $L(\Gamma_0)$ is the class of a measure which is absolutely continuous with respect to the product measure;*
- (ii) *if Γ_0 is torsion-free, then the left–right measure of $L(\Gamma_0)$ is the class of product measure. Moreover, Γ is i.c.c. (infinite conjugacy class) and $\text{Puk}_{L(\Gamma)}(L(\Gamma_0)) = \{m\}$, $m \in \mathbb{N} \cup \{\infty\}$ and Γ_0 is malnormal in Γ .*

Proof. We first prove the statements regarding the left–right measures in both cases and then prove the remaining statements of (ii). In [15, Theorem 3.5], it was shown that the hypothesis is equivalent to the following condition: (ST) For every finite subset $F \subset \Gamma \setminus \Gamma_0$, there exists a finite subset E of Γ_0 such that $gg_0h \notin \Gamma_0$ for all $g_0 \in \Gamma_0 \setminus E$ and all $g, h \in F$.

Let $\widehat{\Gamma}_0$ denote the Pontryagin dual of Γ_0 and let $\lambda_{\widehat{\Gamma}_0}$ denote the normalized Haar measure on $\widehat{\Gamma}_0$. The left–right measure of $L(\Gamma_0)$ is naturally supported on $\widehat{\Gamma}_0 \times \widehat{\Gamma}_0$. Let $u_g \in L(\Gamma)$ be the unitary operator corresponding to the group element $g \in \Gamma$. Fix $g \in \Gamma \setminus \Gamma_0$. Then, taking $F = \{g, g^{-1}\}$, there is a finite subset E of Γ_0 such that $\mathbb{E}_{L(\Gamma_0)}(u_g u_h u_g^*) = 0$ for all $h \in \Gamma_0 \setminus E$. Therefore, by arguments similar to those in the proof of Theorem 4.4 (using, in particular, the appropriate analogue of equation (8)), we get

$$\eta_{u_g}^t(1 \otimes \check{h}) = 0 \text{ for } \lambda_{\widehat{\Gamma}_0} \text{ almost all } t \in \widehat{\Gamma}_0 \text{ and } h \in \Gamma_0 \setminus E,$$

where \check{h} is the canonical image of h in $C(\widehat{\Gamma}_0)$. Recall that $\{u_h : h \in \Gamma_0\}$ is an orthonormal basis of $\ell^2(\Gamma_0)$. Thus, one has

$$\sum_{h \in \Gamma_0} \|\mathbb{E}_{L(\Gamma_0)}(u_g u_h u_g^*)\|_2^2 < \infty.$$

From the proof of Proposition 2.4 of [23] and the remark following it, we get that $\tilde{\eta}_{u_g}^t \ll \lambda_{\widehat{\Gamma}_0}$ and the Radon–Nikodym derivative f_t of $\tilde{\eta}_{u_g}^t$ with respect to $\lambda_{\widehat{\Gamma}_0}$ is in $L^2(\lambda_{\widehat{\Gamma}_0})$ for $\lambda_{\widehat{\Gamma}_0}$ almost all t . However, since $\{\check{h} : h \in \Gamma_0\}$ is an orthonormal basis of $L^2(\lambda_{\widehat{\Gamma}_0})$, one has $f_t = \sum_{h \in \Gamma_0} \langle f_t, \check{h} \rangle \check{h}$ for $\lambda_{\widehat{\Gamma}_0}$ almost all t and the series converge in $L^2(\lambda_{\widehat{\Gamma}_0})$. Consequently, only finitely many terms of the Fourier series survive and hence f_t is continuous for $\lambda_{\widehat{\Gamma}_0}$ almost all t .

If Γ_0 is torsion-free, then so is $\Gamma_0 \times \Gamma_0$. By Lemma 5.6 of [12], η_{u_g} is the normalized Haar measure of the subgroup K_g° , where $K_g = \{(h_1, h_2) \in \Gamma_0 \times \Gamma_0 : h_1 g h_2 = g\}$ and $K_g^\circ = \{\gamma \in \widehat{\Gamma}_0 \times \widehat{\Gamma}_0 : \gamma(K_g) = 1\}$. By the first part of the argument $\eta_{u_g} \ll \lambda_{\widehat{\Gamma}_0} \otimes \lambda_{\widehat{\Gamma}_0}$. Thus, by Lemma 4.6 it follows that $K_g^\circ = \widehat{\Gamma}_0 \times \widehat{\Gamma}_0$, i.e., K_g is trivial.

No matter what Γ_0 be, if $g_i \in \Gamma \setminus \Gamma_0$ and $c_i \in \mathbb{C}$ for $1 \leq i \leq n$, then $\eta_{\sum_{i=1}^n c_i u_{g_i}} = \sum_{i=1}^n |c_i|^2 \eta_{u_{g_i}} + \sum_{i \neq j=1}^n c_i \bar{c}_j \eta_{u_{g_i}, u_{g_j}} \ll \lambda_{\widehat{\Gamma}_0} \otimes \lambda_{\widehat{\Gamma}_0}$ from equation (4). Now, the linear span of

$\{u_g : g \in \Gamma \setminus \Gamma_0\}$ is dense in $\ell^2(\Gamma_0)^\perp$ in $\|\cdot\|_2$. Use Lemma 3.9, 3.10 of [22] to conclude that $\eta_\zeta \ll \lambda_{\widehat{\Gamma_0}} \otimes \lambda_{\widehat{\Gamma_0}}$ for all $\zeta \in \ell^2(\Gamma_0)^\perp$. Finally, use Lemma 5.7 [12] to conclude about the left-right measure.

The statement regarding the Pukánzsky invariant in case (ii) follows directly from Lemma 5.7 [12] and the preceding arguments, as $K_g = K_{g'}$ for all $g, g' \in \Gamma \setminus \Gamma_0$. (The same can be directly deduced from Theorem 4.1 [34] as well by considering the double coset structure of Γ_0 in Γ). Malnormality of Γ_0 is true for the same reason (K_g is trivial for $g \notin \Gamma_0$).

We now show that $L(\Gamma)$ is a factor (in case (ii) of the statement), which will force Γ to be i.c.c. If $p \neq 0$ is a central projection of $L(\Gamma)$, then $L(\Gamma) = L(\Gamma)p \oplus L(\Gamma)(1-p)$. Note that $p \in L(\Gamma_0)$. The left-right measure of $L(\Gamma_0)$ is $[\lambda_{\widehat{\Gamma_0}} \otimes \lambda_{\widehat{\Gamma_0}}]$ and $[\eta_{u_g}] = [\lambda_{\widehat{\Gamma_0}} \otimes \lambda_{\widehat{\Gamma_0}}]$ for $g \in \Gamma \setminus \Gamma_0$. But for any $a, b \in L(\Gamma_0)$, we have

$$\begin{aligned} \langle au_gb, u_g \rangle &= \langle (ap \oplus a(1-p))(u_gp \oplus u_g(1-p))(bp \oplus b(1-p)), (u_gp \oplus u_g(1-p)) \rangle \\ &= \langle apu_gp b p, u_gp \rangle + \langle a(1-p)u_g(1-p)b(1-p), u_g(1-p) \rangle. \end{aligned}$$

This shows that η_{u_g} is supported on the union of two measurable rectangles and hence the left-right measure is concentrated on the same set. Consequently, the left-right measure cannot be equal to the product class unless $p = 1$. Thus, $L(\Gamma)$ is a factor. \square

Remark 4.8. As was pointed out to us by the referee of an earlier version of this paper, (ii) of Theorem 4.7 does not hold if Γ_0 has torsion. For example, consider the inclusion $\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subset \mathbb{F}_2 \times \mathbb{Z}/n\mathbb{Z}$, where \mathbb{Z} is a free factor of \mathbb{F}_2 . Then the hypothesis of Theorem 4.7 is satisfied but the left-right measure of the inclusion is absolutely continuous but not Lebesgue, as $\mathbb{F}_2 \times \mathbb{Z}/n\mathbb{Z}$ is not i.c.c. Also, in the torsion-free case malnormality of Γ_0 is equivalent to strong mixing, for malnormality forces the masa to be of product class. Malnormal subgroups were used by Popa to gain control over normalizers and relative commutants [27]. Recently, Robertson and Steger [31] have proved that, if G is a connected semisimple real algebraic group such that $G(\mathbb{R})$ has no compact factors, then any torsion-free uniform lattice subgroup Γ of $G(\mathbb{R})$ contains a malnormal abelian subgroup Γ_0 such that $Puk_{L(\Gamma)}(L(\Gamma_0)) = \{\infty\}$. Thus, the group von Neumann factor of any such torsion-free uniform lattice subgroup contains a singular masa of product class and infinite multiplicity. In general, it is of interest to know whether every II_1 factor has a singular masa of product class and infinite multiplicity. It is also to be noted that, there are no examples so far of masas in II_1 factors for which the left-right measure is absolutely continuous with respect to the product class, but not equivalent to the product class.

The significance of the next corollary will become clear in the next section, when we relate the left-right measure to the maximal spectral type of an action.

Corollary 4.9. *There does not exist any countable discrete group of the form $\Gamma \rtimes_\alpha \Gamma_0$ with $|\Gamma| = \infty$, and Γ_0 being torsion-free and abelian, such that $L(\Gamma_0) \subset L(\Gamma) \rtimes_\alpha \Gamma_0$ is a mixing masa for which the left-right measure is absolutely continuous with respect to the product measure but not equivalent to the product measure.*

The next result is in spirit similar to the results in [21] regarding wandering vectors. The results in [21] are statements about modules over abelian von Neumann algebras, while the next result deals with bimodules. Theorem 4.10 precisely generalizes the ‘malnormality condition’ (in the context of group inclusions) for masas of product class.

Theorem 4.10. *Let $A \subset M$ be a masa of product class. Let $v \in A$ be a Haar unitary generator of A . Let*

$$W(v) = \{\zeta \in L^2(M) \ominus L^2(A) : \mathbb{E}_A(\zeta v^n \zeta^*) = 0, \text{ for all } n \neq 0\}.$$

Then $\text{span } W(v)$ is dense in $L^2(A)^\perp$.

Proof. Note that for $\zeta \in L^2(M) \ominus L^2(A)$, if $\mathbb{E}_A(\zeta v^n \zeta^*) = 0$ for all $n \neq 0$ for some Haar unitary generator, then the same is true for any Haar unitary generator (see discussion after Theorem 2.1 [23]).

Let $\mathcal{E} = \{\underline{\epsilon} = \{\epsilon_{i,n}\}_{0 \leq i < n, n \in \text{Puk}(A)} : \epsilon_{i,n} = \pm 1\}$. Without loss of generality, let v correspond to the function $t \mapsto e^{2\pi i t}$. The left–right measure of A is $[\lambda \otimes \lambda]$. For $\mathbb{N} \cup \{\infty\} \ni n \in \text{Puk}(A)$ there exist vectors $\zeta_i^{(n)}$, $0 \leq i < n$, so that the projections $P_i^{(n)} : L^2(M) \rightarrow \overline{A\zeta_i^{(n)}A}^{\|\cdot\|_2}$ are mutually orthogonal, equivalent in \mathcal{A}' , $\overline{A\zeta_i^{(n)}A}^{\|\cdot\|_2} \perp L^2(A)$, $\mathcal{A}'(\sum_{0 \leq i < n} P_i^{(n)})$ is the type I_n central summand of $\mathcal{A}'(1 - e_A)$, and, for $a, b \in C[0, 1] \subset A$ and for all $\underline{\epsilon} \in \mathcal{E}$,

$$\left\langle a \left(\bigoplus_{n \in \text{Puk}(A)} \left(\bigoplus_{0 \leq i < n} \epsilon_{i,n} \zeta_i^{(n)} \right) \right) b, \bigoplus_{n \in \text{Puk}(A)} \left(\bigoplus_{0 \leq i < n} \epsilon_{i,n} \zeta_i^{(n)} \right) \right\rangle = \int_{[0,1] \times [0,1]} a(t)b(s) d\lambda(t) d\lambda(s).$$

Fix $\underline{\epsilon} \in \mathcal{E}$ and let $\zeta_{\underline{\epsilon}} = \bigoplus_{n \in \text{Puk}(A)} \bigoplus_{0 \leq i < n} \epsilon_{i,n} \zeta_i^{(n)}$. By Lemma 2.3, we find

$$(9) \quad \|\mathbb{E}_A(\zeta_{\underline{\epsilon}} v^n \zeta_{\underline{\epsilon}}^*)\|_1 = \int_0^1 |\lambda(1 \otimes v^n)| d\lambda(t) = 0, \text{ for all } n \neq 0.$$

Note that $\zeta_{\underline{\epsilon}} \perp L^2(A)$. For $u \in \mathcal{U}(A)$ and $b, c \in A$, use equation (9) to conclude that

$$\mathbb{E}_A((b\zeta_{\underline{\epsilon}}u)v^n(c\zeta_{\underline{\epsilon}}u)^*) = b\mathbb{E}_A(\zeta_{\underline{\epsilon}}v^n\zeta_{\underline{\epsilon}}^*)c^* = 0 \text{ for all } n \neq 0.$$

Let

$$W = \text{span } \{b\zeta_{\underline{\epsilon}}u : u \in \mathcal{U}(A), b \in A, \underline{\epsilon} \in \mathcal{E}\}.$$

It is easy to check that W is dense in $L^2(A)^\perp$. □

5. MASAS FROM DYNAMICAL SYSTEMS

We begin this section with a remark about mixing measures. By the Riemann–Lebesgue Lemma, any measure absolutely continuous with respect to the Lebesgue measure is mixing; however, there are many mixing singular measures as well. Any measure absolutely continuous with respect to a mixing measure is mixing. Thus, mixing is a property of equivalence classes of measures. Mixing measures can be characterized in a geometric way as being asymptotically uniformly distributed [17, Proposition 2.6]. In this section, we will analyze the notion of mixing masas from a spectral theory point of view.

In the next theorem, we relate mixing actions of countable discrete abelian groups to Fourier coefficients of the left–right measures of the associated mixing masas in the crossed product constructions. For simplicity, we work with \mathbb{Z} –actions.

Theorem 5.1. *Let α be a free mixing action of \mathbb{Z} on a diffuse separable finite von Neumann algebra N preserving a faithful normal tracial state τ . If $[\eta]$ is the left–right measure of $L(\mathbb{Z}) \subset N \rtimes_{\alpha} \mathbb{Z}$, then $\tilde{\eta}^t$ is a mixing measure for λ almost all t .*

Proof. Let $M = N \rtimes_{\alpha} \mathbb{Z}$. The tracial state on M will be denoted by τ as well. Let $u_n \in M$ be the canonical unitaries implementing the action. Suppose $x \in N$ and $n, n_1, n_2 \in \mathbb{Z}$. Then, the equation

$$(10) \quad \langle u_{n_1} x u_n u_{n_2}, x u_n \rangle = \tau(u_{n_1} x u_n u_{n_2} u_{-n} x^*) = \tau(u_{n_1} x u_{n_2} x^*),$$

implies that $\eta_{x u_n} = \eta_x$ for all $x \in N$ and all $n \in \mathbb{Z}$.

Note that the left-right measure of $L(\mathbb{Z}) \subset M$ is naturally supported on $\widehat{\mathbb{Z}} = S^1$. Identify $L(\mathbb{Z}) = L^\infty(S^1, \lambda)$, where λ is the normalized Haar measure on S^1 , via the standard identification which sends u_n to the function $e_n(t) = t^n$, $t \in S^1$, $n \in \mathbb{Z}$. Now, for $x \in N$ and $m \in \mathbb{Z}$,

$$\mathbb{E}_{L(\mathbb{Z})}(x u_m x^*) = \mathbb{E}_{L(\mathbb{Z})}(x \alpha_m(x^*) u_m) = \mathbb{E}_{L(\mathbb{Z})}(x \alpha_m(x^*)) u_m = \tau(x \alpha_m(x^*)) u_m.$$

Therefore, from Lemma 2.3, $\eta_x^t(1 \otimes e_m) = \tau(x \alpha_m(x^*)) e_m(t)$ for λ almost all $t \in S^1$ and for all m . Since the action α is mixing, so $\tilde{\eta}_x^t$ is a mixing measure for λ almost all t whenever $\tau(x) = 0$.

Let $x = \sum_{i=1}^n x_i u_{k_i} \in M$ be such that $\mathbb{E}_{L(\mathbb{Z})}(x) = 0$ and $k_i \neq k_j$ for $i \neq j$. Therefore, $\tau(x_i) = 0$, for all $1 \leq i \leq n$. Now from equation (10), we get $\eta_x = \sum_{i=1}^n \eta_{x_i} + \sum_{i \neq j=1}^n \eta_{x_i u_{k_i}, x_j u_{k_j}}$. It is easy to see that $d\eta_{x_i u_{k_i}, x_j u_{k_j}} = (1 \otimes e_{k_i - k_j}) d\eta_{x_i, x_j}$ for all $i \neq j$. Thus, from equation (3) and Lemma 3.6 [22] it follows that,

$$\int_{S^1} s^m d\tilde{\eta}_{x_i u_{k_i}, x_j u_{k_j}}^t(s) = \int_{S^1} s^m s^{k_i - k_j} d\tilde{\eta}_{x_i, x_j}^t(s) \rightarrow 0 \text{ as } m \rightarrow \infty$$

for λ almost all t . This shows that $\tilde{\eta}_x^t$ is a mixing measure for λ almost all t .

There is a unit vector $\zeta \in L^2(M) \ominus L^2(L(\mathbb{Z}))$ such that $\eta = \eta_\zeta$. Let $x_n = \sum_{i=1}^{k_n} x_i^{(n)} u_{k_i}^{(n)} \in M$ with $x_i^{(n)} \in N$ be such that $\mathbb{E}_A(x_n) = 0$, $\|x_n\|_2 \leq 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow \zeta$ as $n \rightarrow \infty$ in $\|\cdot\|_2$. Then, $\eta_{x_n} \rightarrow \eta_\zeta = \eta$ in $\|\cdot\|_{t,v}$ from Lemma 3.10 [22]. Then from Lemma 3.9 [22], there is a subsequence n_k with $n_k < n_{k+1}$ for all k and a set $E \subset S^1$ with $\lambda(E) = 0$, such that for all $t \in E^c$, $\tilde{\eta}_{x_{n_k}}^t, \tilde{\eta}^t$ are finite and

$$\sup_{A \subseteq S^1, A \text{ Borel}} \left| \tilde{\eta}_{x_{n_k}}^t(A) - \tilde{\eta}^t(A) \right| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Note that $\tilde{\eta}_{x_{n_k}}^t$ are mixing measures for all k and for λ almost all t . From standard approximation arguments, it follows that $\tilde{\eta}^t$ is a mixing measure for λ almost all t . \square

We now study relations between the spectral measure of an action and the left-right measure of a masa that arises from a dynamical system. Before doing so, we need some preparation on unitary representations. Let H be a locally compact abelian (LCA) group. Note that \widehat{H} is also a LCA group, where \widehat{H} denotes the Pontryagin dual of H . Also note that, if H is discrete, then \widehat{H} is compact and vice versa. Let $\sigma_{\widehat{H}}$ denote the Borel σ -algebra of \widehat{H} . The following result was proved by Stone for the case $H = \mathbb{R}$ and then independently generalized by Naimark, Ambrose and Godement, and is called the SNAG theorem [2, Theorem D.3.1].

Theorem 5.2 (SNAG Theorem). *Let (π, \mathcal{H}) be a strongly continuous unitary representation of a LCA group H on a separable Hilbert space \mathcal{H} . Then there exists a unique regular projection valued measure $E_\pi : \sigma_{\widehat{H}} \rightarrow \text{Proj}(\mathcal{H})$ on \widehat{H} such that*

$$\pi(g) = \int_{\widehat{H}} \overline{\chi(g)} dE_\pi(\chi), \text{ for all } g \in H.$$

Let (X, ν) be a Lebesgue probability space, where X is a compact metrizable space. Let H be a countable discrete abelian group and let α be an automorphic (free) ergodic action of H on X preserving the measure ν . This gives rise to a canonical unitary representation $\pi : H \rightarrow \mathbf{B}(L^2(X, \nu) \ominus \mathbb{C}1)$. Let μ be the maximal spectral type of this representation that arises from a vector $f_0 \in L^2(X, \nu) \ominus \mathbb{C}1$ [24, p. 13]. Consequently, by SNAG theorem and Hahn–Hellinger theorem [24], there is a μ -measurable field of Hilbert spaces $\{\mathcal{H}_\psi\}_{\psi \in \widehat{H}}$ such that

$$(11) \quad L^2(X, \nu) \ominus \mathbb{C}1 \cong \int_{\widehat{H}}^{\oplus} \mathcal{H}_\psi d\mu(\psi)$$

and $L^\infty(\widehat{H}, \mu)$ is unitarily equivalent to the algebra of diagonalizable operators with respect to the decomposition in equation (11). For $f, g \in L^2(X, \nu) \ominus \mathbb{C}1$, denote by $\mu_{f,g}$ the (possibly) complex Borel measure on \widehat{H} obtained as $\mu_{f,g}(B) = \langle E_\pi(B)f, g \rangle$, where $B \subseteq \widehat{H}$ is Borel. We will also denote $\mu_{f,f}$ by μ_f . Thus $\mu_{f_0} = \mu$. The dimension function of this decomposition in equation (11) is the *spectral multiplicity* of the representation π . Now (see Proposition 11 p. 174, [9])

$$(12) \quad (L^2(X, \nu) \ominus \mathbb{C}1) \otimes L^2(\widehat{H}, \lambda_{\widehat{H}}) \cong \int_{\widehat{H} \times \widehat{H}}^{\oplus} \mathcal{H}_{\psi, \chi} d\mu(\psi) d\lambda_{\widehat{H}}(\chi), \text{ where } \mathcal{H}_{\psi, \chi} = \mathcal{H}_\psi,$$

and $\lambda_{\widehat{H}}$ is the normalized Haar measure on \widehat{H} .

Each $h \in H$ defines a continuous function $\widehat{h} : \widehat{H} \rightarrow \mathbb{C}$ by $\widehat{h}(\chi) = \chi(h)$, $\chi \in \widehat{H}$. Furthermore, $h \in H$ defines a unitary operator $m_{\widehat{h}}$ on $L^2(\widehat{H}, \lambda_{\widehat{H}})$ given by $m_{\widehat{h}}(f) = \widehat{h}f$, $f \in L^2(\widehat{H}, \lambda_{\widehat{H}})$, and a projection $e_{\widehat{h}}$ projecting onto $\mathbb{C}\widehat{h}$. Via the Fourier transform, the crossed product factor $R_\alpha = L^\infty(X, \nu) \rtimes_\alpha H$ (which of course is the hyperfinite II₁ factor) is generated by $\{\sum_{h \in H} \alpha_h(f) \otimes e_{\widehat{h}} : f \in L^\infty(X, \nu), h \in H\}$ and $\{1 \otimes m_{\widehat{h}^{-1}} : h \in H\}$ on $L^2(X, \nu) \otimes L^2(\widehat{H}, \lambda_{\widehat{H}})$. Note that we follow [2] for the definition of Fourier transform. One considers this standard (GNS) representation of R_α on $L^2(X, \nu) \otimes L^2(\widehat{H}, \lambda_{\widehat{H}})$. Let J denote the conjugation operator on the space $L^2(X, \nu) \otimes L^2(\widehat{H}, \lambda_{\widehat{H}})$. For $g \in H$, let v_g be the unitary in $\mathbf{B}(L^2(X, \nu))$ that implements the automorphism α_g . That is, if T_g is the measure preserving transformation such that $\alpha_g(f) = f \circ T_g^{-1}$, $f \in L^\infty(X, \nu)$, then $v_g a = a \circ T_g^{-1}$ for all $a \in L^2(X, \nu)$. It is easy to see that for $f \in L^\infty(X, \nu)$ and $g \in H$,

$$(13) \quad J \left((1 \otimes m_{\widehat{g}}) \left(\sum_{h \in H} \alpha_h(\overline{f}) \otimes e_{\widehat{h}} \right) \right) = (v_g \otimes m_{\widehat{g}^{-1}}) \left(\sum_{h \in H} \alpha_h(f) \otimes e_{\widehat{h}} \right), \text{ and} \\ J(1 \otimes m_{\widehat{g}})J = (v_g \otimes m_{\widehat{g}^{-1}}).$$

Theorem 5.3. *The left-right measure $[\eta_{|\Delta(\widehat{H})^c}]$ of the masa $L(H) \subset R_\alpha$ is the equivalence class of $S_*(\mu \otimes \lambda_{\widehat{H}})$, where μ is the maximal spectral type of the unitary representation of H on $L^2(X, \nu) \ominus \mathbb{C}1$ that arises from the action α and $S : \widehat{H} \times \widehat{H} \rightarrow \widehat{H} \times \widehat{H}$ is given by $S(\psi, \chi) = (\chi, \chi\psi)$.*

Proof. Let $\tau_{\mathcal{R}_\alpha}$ denote the faithful normal tracial state of \mathcal{R}_α . It is clear that $\widehat{H} \times \widehat{H}$ is the natural space where the left-right measure is to be built. Write $\eta_{|\Delta(\widehat{H})^c} = \eta$. For $f \in L^\infty(X, \nu)$ and $g \in H$, write $\alpha(f) = \sum_{h \in H} \alpha_h(f) \otimes e_{\widehat{h}}$ and $w_g = 1 \otimes m_{\widehat{g}}$. The operator w_g is canonically

identified with the function $\widehat{g} \in C(\widehat{H})$. For $i = 1, 2$, fix $f_i \in L^\infty(X, \nu)$ and $h_i \in H$. Now for $g_1, g_2 \in H$,

$$(14) \quad \langle w_{g_1} \alpha(f_1) w_{h_1} w_{g_2}, \alpha(f_2) w_{h_2} \rangle_{\tau_{R_\alpha}} = \int_{\widehat{H} \times \widehat{H}} \widehat{g}_1(\psi) \widehat{g}_2(\chi) \widehat{h_1 h_2^{-1}}(\chi) d\eta_{\alpha(f_1), \alpha(f_2)}(\psi, \chi).$$

On the other hand,

$$(15) \quad \begin{aligned} & \langle w_{g_1} \alpha(f_1) w_{h_1} w_{g_2}, \alpha(f_2) w_{h_2} \rangle_{\tau_{R_\alpha}} \\ &= \langle w_{g_1} J w_{g_2^{-1}} J(\alpha(f_1) w_{h_1}), (\alpha(f_2) w_{h_2}) \rangle_{\tau_{R_\alpha}} \\ &= \langle (1 \otimes m_{\widehat{g_1 g_2}}) \alpha(v_{g_2^{-1}} f_1) w_{h_1}, \alpha(f_2) w_{h_2} \rangle_{\tau_{R_\alpha}} \text{ (by equation (13))} \\ &= \tau_{R_\alpha} \left(w_{g_1 g_2} \alpha(v_{g_2^{-1}} f_1) w_{h_1 h_2^{-1}} (\alpha(f_2))^* \right) \\ &= \tau_{R_\alpha} \left(w_{g_1 g_2 h_1 h_2^{-1}} \alpha(v_{g_2^{-1} h_1 h_2^{-1}} (f_1)) (\alpha(f_2))^* \right) \\ &= \tau_{R_\alpha} (w_{g_1 g_2 h_1 h_2^{-1}}) \langle v_{g_2^{-1} h_1 h_2^{-1}} f_1, f_2 \rangle_{L^2(X, \nu)} \text{ (by orthogonality of algebras)} \\ &= \int_{\widehat{H}} \chi(g_1 g_2) \chi(h_1 h_2^{-1}) d\lambda_{\widehat{H}}(\chi) \int_{\widehat{H}} \overline{\psi(g_2^{-1} h_1 h_2^{-1})} d\mu_{f_1, f_2}(\psi) \text{ (by SNAG Theorem)} \\ &= \int_{\widehat{H} \times \widehat{H}} \psi(g_2) \overline{\psi(h_1 h_2^{-1})} \chi(g_1) \chi(g_2) \chi(h_1 h_2^{-1}) d\mu_{f_1, f_2}(\psi) d\lambda_{\widehat{H}}(\chi). \end{aligned}$$

Let $S : \widehat{H} \times \widehat{H} \rightarrow \widehat{H} \times \widehat{H}$ be given by $S(\psi, \chi) = (\chi, \chi\psi)$. Note that S is bijective. As discussed before, let $f_0 \in L^2(X, \nu) \ominus \mathbb{C}1$ be the vector such that $\mu_{f_0} = \mu$. Now for $f \in L^\infty(X, \nu)$, from (14) and (15), we have

$$(\mu_f \otimes \lambda_{\widehat{H}}) \circ S^{-1} = \eta_{\alpha(f)}.$$

Let $n \in \mathbb{N}$ and $f_i \in L^\infty(X, \nu)$, $h_i \in H$ for $1 \leq i \leq n$ be such that $\mathbb{E}_{L(H)}(\sum_{i=1}^n \alpha(f_i) w_{h_i}) = 0$. Then, $\int_X f_i d\nu = 0$ and from equation (14) it follows that

$$d\eta_{\sum_{i=1}^n \alpha(f_i) w_{h_i}} = \sum_{i,j=1}^n (1 \otimes \widehat{h_i h_j^{-1}}) d\eta_{\alpha(f_i), \alpha(f_j)}.$$

Thus, $\eta_{\sum_{i=1}^n \alpha(f_i) w_{h_i}} \ll \eta_{\alpha(\sum_{i=1}^n f_i)} = (\mu_{\sum_{i=1}^n f_i} \otimes \lambda_{\widehat{H}}) \circ S^{-1} \ll (\mu_{f_0} \otimes \lambda_{\widehat{H}}) \circ S^{-1}$. Note that $((\mu_{f_0} \otimes \lambda_{\widehat{H}}) \circ S^{-1})(\Delta(\widehat{H})) = 0$. There is a nonzero vector $\zeta \in L^2(R_\alpha) \ominus L^2(\widehat{H}, \lambda_{\widehat{H}})$ such that $\eta_\zeta = \eta$. By an easy approximation argument it follows that $\eta \ll (\mu_{f_0} \otimes \lambda_{\widehat{H}}) \circ S^{-1}$ ([22, Lemma 3.10]). Note that equations (14), (15) extend to functions $f_i \in L^2(X, \nu)$, in particular these equations are valid for $f_1 = f_2 = f_0$. Working similarly with $f_1 = f_0$ and $f_2 = f_0$ in equations (14), (15), one checks that $(\mu_{f_0} \otimes \lambda_{\widehat{H}}) \circ S^{-1} = \eta_{f_0 \otimes 1}$. Thus, from Lemma 5.7 [12] conclude that $[(\mu_{f_0} \otimes \lambda_{\widehat{H}}) \circ S^{-1}] = [\eta]$.

Finally, from equation (12),

$$(16) \quad L^2(R_\alpha) \ominus L^2(\widehat{H}, \lambda_{\widehat{H}}) \cong \int_{\widehat{H} \times \widehat{H}}^{\oplus} \mathcal{H}_{\chi\psi^{-1}} dS_*(\mu \otimes \lambda_{\widehat{H}})(\psi, \chi)$$

and $(L(H) \cup JL(H)J)''(1 - e_{L(H)})$ is diagonalizable with respect to this decomposition. \square

Remark 5.4. Theorem 5.3 will be used in the next section to distinguish mixing masas in the free group factors.

Remark 5.5. It is a long standing open question in ergodic theory that, whether there exists a measure preserving automorphism of a Lebesgue probability space whose maximal spectral type is absolutely continuous but not Lebesgue. There are philosophies that suggest that the answer could go either way. For an excellent account of Koopman-realizable (through a \mathbb{Z} -action) measures and multiplicities check [16]. Observe that Theorem 5.3, Theorem 4.7 and Corollary 4.9 say that such a Koopman-realizable measure does not exist provided we restrict ourselves to a smaller class of dynamical systems, namely, those that arise as semidirect products of groups.

Corollary 5.6. *Let Γ be any countable discrete group such that $\Gamma_0 \leq \text{Aut}(\Gamma)$, where Γ_0 is a countable discrete torsion-free abelian group. Let the canonical action of Γ_0 on $L(\Gamma)$ be mixing. Then the maximal spectral type of the Γ_0 -action is Lebesgue.*

6. MIXING MASAS IN THE FREE GROUP FACTORS

The understanding of singular masas especially in the free group factors is of worth in the subject. For construction of masas in this section, we require substantial techniques from ergodic theory. In this section, we exhibit uncountably many non conjugate mixing masas in the free group factors with Pukánszky invariant $\{1, \infty\}$. The general strategy is to construct suitable masas in finite amenable von Neumann algebras and appeal to a well known result of Dykema regarding free products [10].

Recall that the rank of a measure preserving automorphism on a standard probability space is greater than or equal to the spectral multiplicity of the associated Koopman operator [24, p. 31]. Thus, if the rank of a mixing automorphism is one, then the spectral multiplicity of the associated Koopman operator must also be one. In the previous section, we have related the spectral multiplicity of a transformation to the Pukánszky invariant of the associated masa (along the direction of the group) in the group measure space construction.

A rank one measure preserving transformation T of the unit interval $[0, 1]$ is constructed by the method of cutting and stacking [13, 20, 24]. These are transformations which admit a sequence of *Rokhlin towers* generating the entire σ -algebra. We will assume that the reader is familiar with the notion of cutting and stacking. The *classical staircase transformation* is one in which, at the k -th stage, one divides the $(k - 1)$ -th stack into k equal columns and put j spacers over the j -th column, $1 \leq j \leq k$, which is why it is called a staircase [24, p. 153]. The next result will be used in constructing masas in the free group factors, but it is also of some independent interest as well.

Theorem 6.1. *There exists a mixing masa B in the hyperfinite II₁ factor R whose Pukánszky invariant is $\{1\}$ and whose left-right measure is singular.*

Proof. The classical staircase automorphism T is mixing and of rank one [1, p. 744]. Consequently, $L(\mathbb{Z}) \subset R_T$ is a mixing masa in the hyperfinite II₁ factor $R_T = L^\infty([0, 1], \lambda) \rtimes_T \mathbb{Z}$, where λ is the Lebesgue measure on $[0, 1]$ [15]. The Pukánszky invariant of $L(\mathbb{Z}) \subset R_T$ is $\{1\}$ from equation (16). Write $B = L(\mathbb{Z})$.

The maximal spectral type of the staircase transformation is given by a Reisz product, which is known to be singular [19] (also see p. 154 [24]). Thus from Theorem 5.3, the left-right measure of $B \subset R_T$ is singular to the product class. \square

Theorem 6.2. *Let $k \in \{2, 3, \dots, \infty\}$ and let Γ be any countable discrete group. There exist uncountably many pairwise non conjugate mixing masas in $L(\mathbb{F}_k * \Gamma)$ whose Pukánszky invariant is $\{1, \infty\}$.*

Proof. Let

$$(17) \quad \mathbb{P}_{\mathbb{N}} = \left\{ \alpha = \{\alpha_n\}_{n=1}^{\infty} : \alpha_n > \alpha_{n+1}, 0 < \alpha_n < 1 \text{ for all } n, \sum_{n=1}^{\infty} \alpha_n = 1 \right\}.$$

For $\alpha, \beta \in \mathbb{P}_{\mathbb{N}}$, say $\alpha \neq \beta$ if $\alpha_n \neq \beta_n$ for some n . Fix $\alpha \in \mathbb{P}_{\mathbb{N}}$.

Let $R_{\alpha} = \bigoplus_{n=1}^{\infty} R$ and $B_{\alpha} = \bigoplus_{n=1}^{\infty} B$, where R and B are as in Theorem 6.1. The projections $p_n = (0 \oplus \cdots \oplus 0 \oplus 1 \oplus 0 \oplus \cdots)$, where 1 appears at the n -th coordinate, is a central projection of R_{α} and it belongs to B_{α} . Equip R_{α} with the faithful trace

$$\tau_{R_{\alpha}}(\cdot) = \sum_{n=1}^{\infty} \alpha_n \tau_R(\cdot p_n),$$

where τ_R denotes the unique tracial state of R . Then B_{α} is a mixing masa in the hyperfinite algebra R_{α} and the latter is separable. The last statement is a simple application of dominated convergence theorem.

The projections p_n correspond to indicator of measurable subsets $E_n \subset (S^1, \mathbf{m})$, so that $E_n \cap E_m$ is a set of \mathbf{m} measure 0 for all $n \neq m$ (where \mathbf{m} is the normalized Haar measure on S^1). Upon applying appropriate transformations, the left-right measure of $B \subset R$ can be transported to each $E_n \times E_n$, which is denoted by $[\eta_n]$. We also assume $\eta_n(E_n \times E_n) = 1$ for all n .

Consider $(M, \tau_M) = (R_{\alpha}, \tau_{R_{\alpha}}) * (R, \tau_R)$. Then M is isomorphic to $L(\mathbb{F}_2)$ by a well known theorem of Dykema [10]. Note that $B_{\alpha} \subset L(\mathbb{F}_2)$ is a mixing masa by Propositions 6.1, 6.5 [4]. The left-right measure of the inclusion $B_{\alpha} \subset L(\mathbb{F}_2)$ is

$$[\mathbf{m} \otimes \mathbf{m} + \sum_{n=1}^{\infty} \frac{1}{2^n} \eta_n]$$

and $\text{Puk}_{L(\mathbb{F}_2)}(B_{\alpha}) = \{1, \infty\}$ from Proposition 5.10 and Theorem 3.2 [12] (also see [23]).

Since automorphisms of II_1 factors are trace preserving, the non conjugacy of B_{α} and B_{β} in $L(\mathbb{F}_2)$ for $\alpha \neq \beta$ follows clearly by considering their left-right measures in the measure-multiplicity invariant.

There exist isomorphisms [10]

$$L(\mathbb{F}_2) * L(\mathbb{F}_{k-2} * \Gamma) \cong L(\mathbb{F}_k * \Gamma) \text{ for } k \geq 2.$$

For $k \geq 2$, each B_{α} is a mixing masa (Propositions 6.1, 6.5 [4]) in $L(\mathbb{F}_k * \Gamma)$ with $\text{Puk}_{L(\mathbb{F}_k * \Gamma)}(B_{\alpha}) = \{1, \infty\}$ [12]. Use Lemma 5.7, Proposition 5.10 [12] to decide the non conjugacy of B_{α} and B_{β} when $\alpha \neq \beta$ in the free product. \square

Remark 6.3. It is difficult to distinguish between two mixing masas in the free group factors. If two masas in a free group factor are of product class, then it becomes a significantly difficult problem (e.g., the conjugacy of the Laplacian masa and the generator masas is a challenging problem [11]). Maximal injectivity of masas can be used but that too in very limited cases. The left-right measure of any masa in the free group factors always contains a part of the product measure as a summand [39]. This statement is one of the deep results in the subject. So, the singular summand of the left-right measure is a plausible candidate that can distinguish two masas with same Pukánszky invariant. It is a very common idea to build a masa in a free group factor by starting with a masa in the hyperfinite II_1 factor. But mixing masas in the hyperfinite II_1 factor arising from ergodic group actions are also rare. The set of mixing

transformations is meager in the (Polish) weak topology on the group of all measure preserving transformations. In [38], a Polish topology strictly stronger than the induced weak topology was introduced and it was shown that a generic mixing transformation has multiplicity $\{1\}$. Nevertheless, many more values of the multiplicity function of mixing transformations were obtained by Danilenko in [7]. But it is yet not known whether the maximal spectral types of these transformations in [7] are singular to Lebesgue measure. In case they are, our technique applies to construct more mixing masas in the free group factors.

APPENDIX A. A TECHNICAL RESULT

As before, let A be a masa in a II₁ factor M . We continue to assume that $A \cong L^\infty([0, 1], \lambda)$, where λ is the Lebesgue measure. The next Lemma was remarked in [23] under a stronger hypothesis. We assume the theory of L^1 spaces associated to finite von Neumann algebras for which we refer the reader to [32].

Lemma A.1. *Let the left-right measure of A be the class of product measure. Let $v \in A$ be the Haar unitary generator corresponding to the function $[0, 1] \ni t \mapsto e^{2\pi i t}$. Then the following are equivalent.*

(i) *There is a set $S \subset L^2(M) \ominus L^2(A)$ such that $\text{span } S$ is dense in $L^2(M) \ominus L^2(A)$, and*

$$\sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(\zeta v^k \zeta^*)\|_2^2 < \infty \text{ for all } \zeta \in S.$$

(ii) *There is a set $S' \subset L^2(M) \ominus L^2(A)$ such that S' is dense in $L^2(M) \ominus L^2(A)$, and*

$$\sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(\zeta_1 v^k \zeta_2^*)\|_2^2 < \infty \text{ for all } \zeta_1, \zeta_2 \in S'.$$

The vectors $\mathbb{E}_A(\zeta_1 v^k \zeta_2^*), \mathbb{E}_A(\zeta v^k \zeta^*)$ in the statement of the above lemma are in $L^1(M)$. But in the statement of Lemma A.1, it is implicit that the sets S, S' can be chosen so that $\mathbb{E}_A(\zeta_1 v^k \zeta_2^*), \mathbb{E}_A(\zeta v^k \zeta^*) \in L^2(A)$. Thus, there is no confusion in considering their L^2 -norms.

Proof. We have to prove (i) \Rightarrow (ii) only. For $b \in C[0, 1]$ and $\zeta \in S$, the Fourier series expansion of b and a simple application of Cauchy-Schwarz inequality show that $\mathbb{E}_A(\zeta b \zeta^*) \in L^2(A)$. Indeed, $b = \sum_{k \in \mathbb{Z}} \langle b, v^k \rangle v^k$ with convergence in $\|\cdot\|_2$. Thus,

$$\begin{aligned} \sup_{a \in A: \|a\|_2 \leq 1} |\tau(\mathbb{E}_A(\zeta b \zeta^*)a)| &\leq \sup_{a \in A: \|a\|_2 \leq 1} \sum_{k \in \mathbb{Z}} |\tau(\mathbb{E}_A(\zeta v^k \zeta^*)a)| |\tau(bv^{-k})| \\ &\leq \sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(\zeta v^k \zeta^*)\|_2 |\tau(bv^{-k})| \\ &\leq \left(\sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(\zeta v^k \zeta^*)\|_2^2 \right)^{\frac{1}{2}} \|b\|_2 < \infty. \end{aligned}$$

This proves the claim.

From Lemma 5.7 [12], we have $\eta_\zeta \ll \lambda \otimes \lambda$. Then Lemma 2.3 and standard theory of Fourier series show that:

(a) $f_\zeta = \frac{d\eta_\zeta}{d(\lambda \otimes \lambda)}$ is in $L^2(\lambda \otimes \lambda)$ for all $\zeta \in S$,

(b) $\|\mathbb{E}_A(\zeta b \zeta^*)\|_2^2 = \int_0^1 |\lambda(f_\zeta(t, \cdot)b)|^2 d\lambda(t)$ for all $b \in C[0, 1]$.

To prove (a) observe that

$$\int_0^1 \sum_{k \in \mathbb{Z}} |\eta_\zeta^t(1 \otimes v^k)|^2 d\lambda(t) = \sum_{k \in \mathbb{Z}} \int_0^1 |\eta_\zeta^t(1 \otimes v^k)|^2 d\lambda(t) = \sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(\zeta v^k \zeta^*)\|_2^2 < \infty.$$

Thus, for λ almost all t one has,

$$\sum_{k \in \mathbb{Z}} \left| \int_0^1 f_\zeta(t, s) v^k(s) d\lambda(s) \right|^2 < \infty.$$

Thus (a) is established upon using Lemma 3.6 [22]. The proof of (b) is an easy consequence of (a) and Theorem 2.3.

Let $\xi_1 = \sum_{i=1}^n c_i \zeta_i^1$, $\xi_2 = \sum_{j=1}^m d_j \zeta_j^2$ with $\zeta_i^1, \zeta_j^2 \in S$, $c_i, d_j \in \mathbb{C}$ for all $1 \leq i \leq n$, $1 \leq j \leq m$. Note that for all i, j ,

$$\sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(\zeta_i^1 v^k \zeta_i^{1*})\|_2^2 < \infty, \sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(\zeta_j^2 v^k \zeta_j^{2*})\|_2^2 < \infty.$$

Then by assumption $f_{\zeta_i^1}, f_{\zeta_j^2} \in L^2(\lambda \otimes \lambda)$ for all i, j . From equation (4), $\eta_{\zeta_i^1, \zeta_j^2} \ll \lambda \otimes \lambda$. But because $|\eta_{\zeta_i^1, \zeta_j^2}| \leq \eta_{\zeta_i^1} + \eta_{\zeta_j^2}$, we conclude that $f_{\xi_1, \xi_2} = \frac{d\eta_{\xi_1, \xi_2}}{d(\lambda \otimes \lambda)} \in L^2(\lambda \otimes \lambda)$.

Fix $a, b \in C[0, 1]$. Then as τ extends to L^1 ,

$$\begin{aligned} (18) \quad \int_0^1 a(t) \mathbb{E}_A(\xi_1 b \xi_2^*)(t) d\lambda(t) &= \tau(a \mathbb{E}_A(\xi_1 b \xi_2^*)) = \tau(a \xi_1 b \xi_2^*) = \int_{[0,1] \times [0,1]} a(t) b(s) d\eta_{\xi_1, \xi_2}(t, s) \\ &= \int_{[0,1] \times [0,1]} a(t) b(s) f_{\xi_1, \xi_2}(t, s) d\lambda(t) d\lambda(s) \\ &= \int_0^1 a(t) \lambda(f_{\xi_1, \xi_2}(t, \cdot) b) d\lambda(t). \end{aligned}$$

Now consider the function $[0, 1] \ni t \xrightarrow{g} \lambda(f_{\xi_1, \xi_2}(t, \cdot) b)$. It is clearly λ -measurable and

$$\begin{aligned} \int_0^1 |\lambda(f_{\xi_1, \xi_2}(t, \cdot) b)|^2 d\lambda(t) &= \int_0^1 \left| \int_0^1 f_{\xi_1, \xi_2}(t, s) b(s) d\lambda(s) \right|^2 d\lambda(t) \\ &\leq \|b\|^2 \int_0^1 \left(\int_0^1 |f_{\xi_1, \xi_2}(t, s)| d\lambda(s) \right)^2 d\lambda(t) \\ &\leq \|b\|^2 \int_0^1 \int_0^1 |f_{\xi_1, \xi_2}(t, s)|^2 d\lambda(t) d\lambda(s) < \infty. \end{aligned}$$

Therefore, from equation (18) we get,

$$\begin{aligned} \sup_{a \in C[0,1], \|a\|_2 \leq 1} \left| \int_0^1 a(t) \mathbb{E}_A(\xi_1 b \xi_2^*)(t) d\lambda(t) \right| &= \sup_{a \in C[0,1], \|a\|_2 \leq 1} \left| \int_0^1 a(t) \lambda(f_{\xi_1, \xi_2}(t, \cdot) b) d\lambda(t) \right| \\ &= \left(\int_0^1 |\lambda(f_{\xi_1, \xi_2}(t, \cdot) b)|^2 d\lambda(t) \right)^{\frac{1}{2}} < \infty. \end{aligned}$$

Thus, it follows that $\mathbb{E}_A(\xi_1 b \xi_2^*) \in L^2(A)$ and

$$\|\mathbb{E}_A(\xi_1 b \xi_2^*)\|_2^2 = \int_0^1 |\lambda(f_{\xi_1, \xi_2}(t, \cdot) b)|^2 d\lambda(t).$$

Consequently, from Lemma 2.3 we have

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(\xi_1 v^k \xi_2^*)\|_2^2 &= \sum_{k \in \mathbb{Z}} \int_0^1 |\eta_{\xi_1, \xi_2}^t(1 \otimes v^k)|^2 d\lambda(t) = \int_0^1 \sum_{k \in \mathbb{Z}} |\eta_{\xi_1, \xi_2}^t(1 \otimes v^k)|^2 d\lambda(t) \\
&= \int_0^1 \sum_{k \in \mathbb{Z}} \left| \int_0^1 f_{\xi_1, \xi_2}(t, s) v^k(s) d\lambda(s) \right|^2 d\lambda(t) \quad (\text{Lemma 3.6 [22]}) \\
&= \int_0^1 \|f_{\xi_1, \xi_2}(t, \cdot)\|_{L^2(\lambda)}^2 d\lambda(t) \\
&= \int_{[0,1] \times [0,1]} |f_{\xi_1, \xi_2}(t, s)|^2 d(\lambda \otimes \lambda)(t, s) < \infty.
\end{aligned}$$

Finally, let $S' = \text{span } S$. □

Remark A.2. When A is a masa of product class, the set S in Lemma A.1 can be chosen such that $\frac{d\eta_\zeta}{d(\lambda \otimes \lambda)}$ is essentially bounded for $\zeta \in S$ (see proof of Theorem 2.5 and 2.7, [23]). So the same is true for vectors in $S' = \text{span } S$.

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